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Integration and Applications in Economic Dynamics

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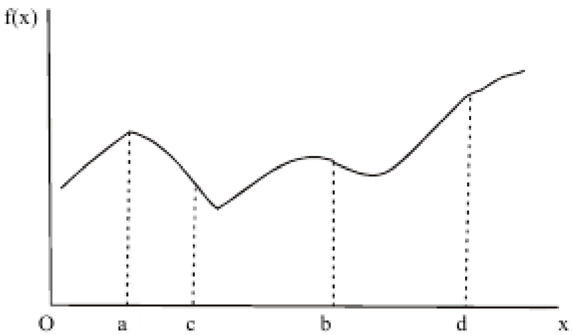
Abstract. In a dynamic economics model, the basic objective is the identification of the time path of the variable on the basis of its rate of change. For example, national income y of a country changes overtime. To see the rate of change we need to see its change with respect to time and to find the time path followed by y. Thus, if we know the derivative $\frac{dy}{dt}$, it will be possible to get onto the function like $y = y(t)$ through the technique of integration which happens to be opposite of the process of differentiation. We will return to this process after a while.
Keywords: Consumer's Surplus, Capital Accumulation Over a Specified Period , Present Value or Discounted Value Under Continuous Compounding of Interest, sum, stream of income, optimal timing.

Introduction: Events that change over time are put under the purview of dynamic analysis. In this unit, we introduce a framework for dealing with dynamic economic problems by introducing time explicitly into these. For that purpose, let us start with the mathematical techniques of integral calculus and differential equations.

The Definite Integral and its Economic Applications

The definite integral of the function $f(x)$ over the interval $[a,b]$ is expressed symbolically as $\int_a^b f(x)dx$, read as "the integral of f with respect to x from a to b". The smaller number a is termed the lower limit and b, the upper limit, of integration. Geometrically, this definite integral denotes the area under the curve representing $f(x)$ between the points $x = a$ and $x = b$.



It should be noted that the indefinite integral $f(x)$ dx is a function of x,

whereas the definite integral $\int_a^b f(x)dx$ is a number. The numerical value of the definite integral depends on the two limits of integral also changes. This is clear from Figure where if we change the interval (a,b) to (c,d) the value of the area under the curve will, in general, change. Another feature of the definite integral is that its value does not depend on the particular symbol chosen to represent the independent variable so long as the form of the function is not changed. That is,

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du = etc.$$

The following theorem establishes the connection between indefinite and definite integration and supplies the method for evaluating definite integrals.

The Fundamental Theorem of Calculus

If $\int_a^b f(x)dx = f(x)+c$, then $\int_a^b f(x)dx = f(b)-f(a)$.

Examples: 1) To evaluate $\int_1^5 x^2dx$.

$$\int x^2dx = \frac{x^3}{3}+C$$

So, $\int_1^5 x^2dx = \frac{5^3}{3}-\frac{1^3}{3}=\frac{124}{3}$.

2) To evaluate $\int_{-1}^1 (ax^2+bx+c)dx$.

$$\int\left(ax^2+bx+c\right)dx = a\frac{1}{3}x^3+b\frac{1}{2}x^2+cx$$

So,

$$\int_{-1}^1\left(ax^2+bx+c\right)dx = \left(a\frac{1^3}{3}+b\frac{1^2}{2}+c\cdot 1\right)-\left(a\frac{(-1)^3}{3}+b\frac{(-1)^2}{2}+c\cdot(-1)\right)=\left(\frac{a}{3}+\frac{b}{2}+c\right)-\left(\frac{-a}{3}+\frac{b}{2}-c\right)=$$

$$=\frac{2a}{3}+2c=2\left(\frac{a}{3}+c\right).$$

Definite integrals are subject to certain rules of operation.

Rule 1: If the two limits are equal, the value of the integral is zero.

$$\int_a^a f(x)dx = 0.$$

Rule 2: Reversing the limits of integration changes the sign of the integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Rule 3: The definite integral can be expressed as the sum of subintegrals.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

where b is a point within the interval (a,c).

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We now discuss briefly one special type of definite integral, the improper integral. When one of the limits of integration is $(+\infty)\, or\, (-\infty)$ a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits according to the following rules:

i) $$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$$

ii) $$\int_{-\infty}^{b} f(x)dx = \lim_{a\to\infty}\int_a^b f(x)dx.$$

Example:

Evaluate $\int_1^{\infty}\frac{dx}{x^2}$

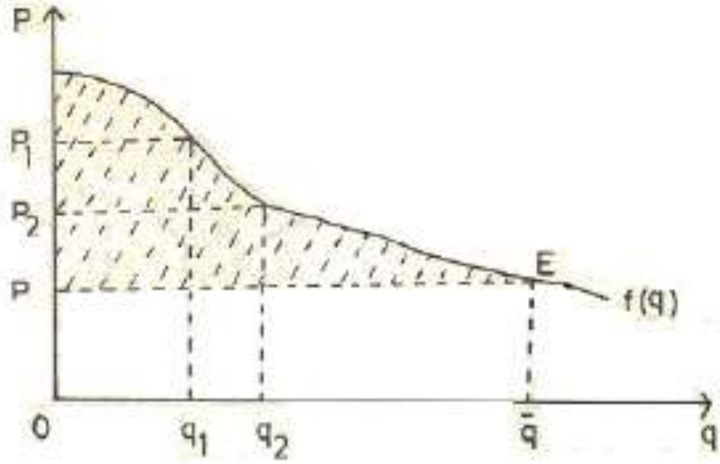
Since $\int_1^b \frac{dx}{x^2} = -\frac{1}{b}+1,$ the desired integral is

$$\lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left(-\frac{1}{b}\right) = 1.$$

Economic Applications of the Definite Integral

a) Consumer's Surplus

Consumer's surplus (CS) measures the net benefit that a consumer enjoys from the purchase of a particular commodity in the market. To measure CS, we take (i) the demand function of a consumer $P = f(q)$ representing the highest price a consumer is willing to pay (her 'demand price') for any specified quantity, (ii) the actual price paid for the quantity purchased and (iii) get the difference between (i) and (ii). In the figure below, a consumer is willing to pay a price of p_1 per unit for q1 units, p_2 per unit for q_2 units, and so on. Suppose the market price is $\bar{p}$. At this prices she purchases $\bar{q}$ units and her actual expenditure is $\overline{pq}$, represented by the rectangle $O\bar{q}E\bar{q}$. Her total willingness to pay for $\bar{q}$ is obtained as the sum of her demand prices for all the units from 0 to $\bar{q}$.



Mathematically, this is the definite integral of the demand function up to $\bar{q}$,or the area under the demand curve up to $\bar{q}$. The excess of this total llingness to pay in units of money over her actual expenditure is her Consumer's surplus:

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$$CS = \int_0^{\bar{q}} f(q)dq - \overline{pq}.$$

It is represented by the crossed area in the diagram.

Example: Suppose the demand function of a consumer is given by $p = 80 - q$.

If the price offered is $p = 60$, find the consumer surplus. For $p = 60$, we get $q = 20$ from the demand equation. Actual expenditure $pq = 1200$.

Now $CS = \int_0^{20} (80 - q)\, dq - pq = 1400 - 1200 = 200.$

Thus the consumer's surplus is Rs.200.

b) Capital Accumulation Over a Specified Period

Since $\int I(t)dt = K(t) + C$, we may use the definite integral $\int_a^b I(t)dt = K(b) - K(a)$ to find the total capital accumulation during the time interval $[a, b]$.

Example: Given the rate of net investment $I(t) = 9t^{\frac{1}{2}}$, find the level of capital formation in (i) 16 years and (ii) between the 4th and the 8th years.

i) $K = \int_0^{16} 9t^{\frac{1}{2}} dt = 6(16)^{\frac{3}{2}} = 384$

ii) $K = \int_4^8 9t^{\frac{1}{2}} dt = 6(8)^{\frac{3}{2}} - 6(4)^{\frac{3}{2}} = 135,76 - 48 = 87,76.$

c) Present Value or Discounted Value Under Continuous Compounding of Interest

A basic concept in capital theory is the present or discounted or capital value of a specified sum of money that will be available at a future date. If the annual rate of interest is 100r percent, then the present value Y of Rs. x available next year is $Y = \frac{x}{1+r}$, because Rs. $\left(\frac{x}{1+r}\right)$ now will become Rs.x after one year at the stipulated annual rate of interest of 100r per cent. Similarly, the present value of Rs. x available t years hence is $Y = \frac{x}{(1+r)^t}$. If interest is compounded n times a year at 100r per cent per year then the present value is

$$Y = \frac{x}{\left(1+\frac{r}{n}\right)^{nt}} = x\left(1+\frac{r}{n}\right)^{nt} \qquad \ldots(1)$$

If interest is compounded continuously, then $n \to \infty$ and the continuous counterpart of (1) becomes $Y = xe^{-rt}$. Using the result: $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^{nx} = e^{kx}$.

Now consider a project that yields an income $x(t)$ at future period t for $t = 1, 2, ..., T.$. That is, the income stream associated with the project for T years is $x(1), x(2), ..., x(T)$. The present or discounted value of this income stream at annual compounded is:

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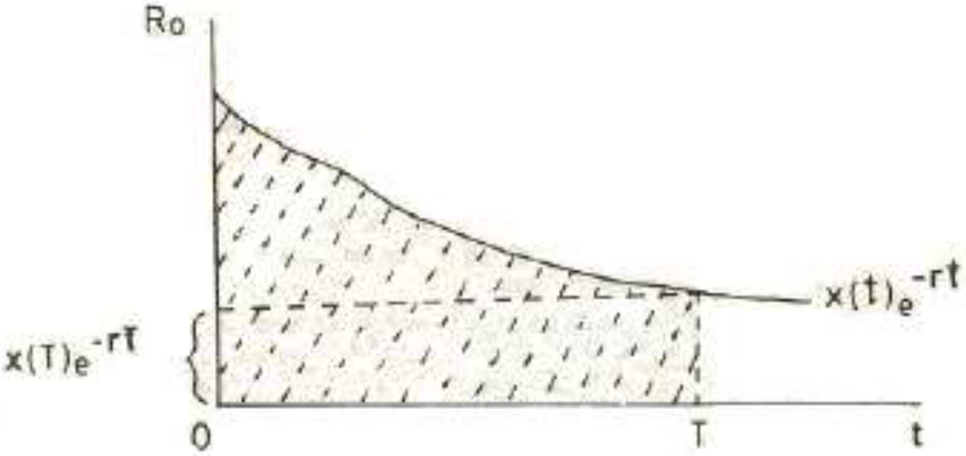
$$Y=\sum_{t=1}^{T}\frac{x(t)}{(1+r)^{t}} \quad \ldots(2)$$

When income flows continuously at the rate of $x(t)$ per period up to period T and interest is compounded continuously the expression for present value becomes

$$Y=\int_{0}^{T}x(t)e^{-rt}dt \quad \ldots(3)$$

Note that the magnitude of present value depends on the size of the income stream, the number of years it flows (the time horizon) and the rate of interest (the discount factor). You should keep in mind the distinction between the resent value of the **sum** $x(T)$ available T periods hence and the present value of the **stream of income** $x(t)$ per period up to period T. In the figure the former is the ordinate at $t=T$, whereas the latter is the shaded area under the curve upto $t=T$.



A particular case of interest is the valuation of an asset (a bond or a piece of land) yielding a fixed income Rs. R for ever. The market value Y of such an asset is the present value of the perpetual yield:

$$Y=\int_{0}^{\infty}\mathrm{Re}^{-rtdt}=R\int_{0}^{\infty}\mathrm{e}^{-rtdt}.$$

Remembering the rule for evaluating improper integrals.

$$\int_{0}^{\infty}\mathrm{e}^{-rtdt}=\lim_{b\to\infty}\int_{0}^{b}e^{-rt}dt=\lim_{b\to\infty}\left(-\frac{1}{r}e^{-rb}+\frac{1}{r}\right)=\frac{1}{r}$$

Hence, the market value is:

$$Y=\frac{R}{r}.$$

To illustrate further, the use of the concept of present value, we consider the following more complex problem of **optimal timing**. The value of timber planted on a plot of land grows over time according to the function $V(t)=2^{\sqrt{t}}$. Assuming zero cost of maintenance and a discount factor of r, find the optimal time to cut the timber for sale. Since cost of production (upkeep) is zero, profit maximisation here is equivalent to the maximisation of sales revenue V. Due to the interest factor, r, different V values, however, are not comparable because they accrue at different points of time. The solution

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involves discounting each V value to its present value (the value at $t=0$). The process of discounting puts them on comparable footing. Assuming continuous compounding, the present value $R(t)$ can be written as $R(t)=V(t)e^{-rt}=2^{\sqrt{t}}e^{-rt}$. The optimal time of cutting is the value of t that maximises $R(t)$. Since $f(x)$ and log $f(x)$ attain their maximum at the same value of x, the problem can equivalently be restated as finding the value of t that maximises log $R(t)$:

$\ln R(t)=\sqrt{t}\log 2-rt$.

Differentiating with respect to t and setting the derivative equal to zero, we get

$$\frac{1}{R}-\frac{dR}{dt}=\left(\frac{\ln 2}{2\sqrt{t}}-r\right)=0 \quad or \quad \frac{dR}{dt}=R\left(\frac{\ln 2}{2\sqrt{t}}-r\right)=0\,(R\neq 0) \quad or \quad \sqrt{t}=\frac{\ln 2}{2r} \quad or \quad t=\left(\frac{\ln 2}{2r}\right)^2$$

We leave it to you to check that at this value of t the second order condition for maximisation $\frac{d^2R}{dt^2}<0$ is also satisfied. Thus, the expression for the optimum time for cutting the timber is $\left(\frac{\ln 2}{2r}\right)^2$. It is to be noted that the higher the rate of discount r, the sooner the timber should be cut. This is a general characteristic of all optimal storage or timing problems.

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