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Some Economic Applications of Differential Equations

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Abstract. Applications of differential equations are now used in modeling motion and change in all areas of science. The theory of differential equations has become an essential tool of economic analysis particularly since computer has become commonly available. It would be difficult to comprehend the contemporary literature of economics if one does not understand basic concepts (such as bifurcations and chaos) and results of modem theory of differential equations.

Keywords: model, stock, prices, bonds, a financial operation, percentages, continuity.

Introduction

Mathematical modeling in economics became central to economic theory during the decade of the Second World War. The leading figure in that period was Paul Anthony Samuelson whose 1947 book, Foundations of Economic Analysis, formalized the problem of dynamic analysis in economics. In this brief chapter some seminal applications of differential equations in economic growth, capital and business trade cycles are outlined in deterministic setting. Chaos and bifurcations in economic dynamics are not considered. Explicit analytical solutions are presented only in relatively straightforward cases and in more complicated cases a path to the solution is outlined. Differential equations in modern dynamic economic modeling are extensions and modifications of these classical works. Finally we would like to stress that the differential equations presented in this chapter are of the "stand-alone" type in that they were solely introduced to model economic growth and trade cycles. Partial differential equations such as those which arise in related fields, like Bioeconomics and Differential Games, from optimizing the Hamiltonian of the problem, and stochastic differential equations of Finance and Macroeconomics are not considered here.

A Model of Stock Prices

Consider a simple financial market structure consisting of only two assets:

(1) government bonds which pay a fixed interest rate, r;

(2) shares of stock, which pay a constant stream of dividends, d.

Taking r and d as given, we want to construct a model to predict the evolution of share prices, p[1].

[1]This model is borrowed from De la Fuente (2000, pp. 503-513).

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A. Bonds

The return on bonds depends on the annual interest rate, r, and on the period of capitalization T (in years).

If capitalization is annual, then, the value at t of an investment, I, made at t_0 is:

$$V_B(t) = I(1+r)^{t-t_0}.$$

With the same interest rate, but monthly capitalization (the monthly interest rate is $r/12$), the value at t of the same investment made at t_0 is:

$$V_B(t) = I\left(1+\frac{r}{12}\right)^{12(t-t_0)}.$$

If capitalization occurs n times per year, the value at t of the investment made at t_0 is:

$$V_B(t) = I\left(1+\frac{r}{n}\right)^{n(t-t_0)}.$$

Finally, with instantaneous capitalization, the value at t of the investment made at t_0 is:

$$V_B(t) = \lim_{n\to+\infty} I\left(1+\frac{r}{n}\right)^{n(t-t_0)} = I\left[\lim_{n\to+\infty}\left(1+\frac{r}{n}\right)^{n}\right]^{t-t_0} = Ie^{r(t-t_0)}.$$

In the continuous time version of the model, we assume that capitalization is instantaneous (takes place in continuous time).

The growth at t of the value of the investment made at t_0 is:

$$V_B'(t) = Ire^{r(t-t_0)}.$$

Growth at t_0 is simply:

$$V_B'(t_0) = Ir.$$

B. Stocks

The return on an investment in stocks has two components:

i) a fixed annual dividend, d, per unit of stock;

ii) a capital gain, due to variations in the stock price.

For simplicity, we assume a continuous flow of dividends.

An investment, I, made at t_0 consists in buying $\frac{I}{p(t_0)}$ units of stock, at the prevailing market price, $p(t_0)$.

The value at t of an investment in stock, I, made at t_0 is:

$$V_S(t) = \frac{I}{p(t_0)}[d(t-t_0) + p(t)].$$

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The growth at t of the value of the investment made at t_0 is:

$$V_S'(t) = \frac{I}{p(t_0)}[d + \dot{p}(t)].$$

Growth at t_0 is:

$$V_S'(t) = \frac{I}{p(t_0)}[d + \dot{p}(t_0)].$$

The present value of the dividend stream is designated as the *fundamental value* of the stock. Notice that it is constant over time.

$$VF_S = \int_0^\infty de^{-rt}dt = \frac{d}{r}. \qquad (1)$$

C. Arbitrage and Equilibrium

Arbitrage is the practice of taking advantage of price differentials between different markets. It consists of a trade that involves no possibility of loss and a positive probability of gain [3].

No-Arbitrage Principle: In equilibrium, there are no arbitrage opportunities that can be explored.

The absence of arbitrage opportunities implies that the two assets (bonds and stocks) have the same expected return, in each moment in time. The only uncertainty concerns the variation in the price of the stock.

No-arbitrage at implies the following dynamic condition:

$$Ir = \frac{I}{p(t_0)}[d + \dot{p}^e(t_0)]. \qquad (2)$$

It must hold for all t_0, therefore, we can write it as:

$$\dot{p}^e(t_0) = rp(t_0) - d. \qquad (3)$$

D. Adaptive Expectations

We still haven't defined the way in which agents form their expectations. In a first approximation, we assume that agents have *adaptive expectations*. If the actual price of the stock exceeds the expected price, agents revise their forecasts in the direction of the observed price [4].

$$\dot{p}^e(t_0) = \alpha[p(t) - p^e(t)]. \qquad (4)$$

Some manipulation yields a differential equation governing expected stock prices.

$$\dot{p}^e(t) = \alpha\left[\frac{d}{r} + \frac{\dot{p}^e(t)}{r} - p^e(t)\right] \quad \Leftrightarrow \quad \dot{p}^e(t) = -\frac{r\alpha}{r-\alpha}p^e(t) + \frac{\alpha d}{r-\alpha}. \qquad (5)$$

To make sure that the system is stable, assume that $\alpha < r$ (expectations are not too volatile).

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The *steady-state* value of p^e, denoted p^{e_*}, can be determined by setting $\dot{p}^e = 0$. It coincides with the fundamental value of the stock.

$$0 = -\frac{r\alpha}{r-\alpha} p^{e_*} + \frac{\alpha d}{r-\alpha} \quad \Leftrightarrow \quad p^{e_*} = \frac{d}{r}. \qquad (6)$$

Solving the differential equation, we find that the evolution of p^e is described by the following expression.

$$p^e(t) = p^{e_*} + [p^e(t_0) - p^{e_*}]e^{-\frac{r\alpha}{r-\alpha}(t-t_0)}$$

$$p^e(t) = \frac{d}{r} + \left[p^e(t_0) - \frac{d}{r} \right] e^{-\frac{r\alpha}{r-\alpha}(t-t_0)}. \qquad (7)$$

The fact that the system is stable means that a discrepancy between the fundamental value of the stock and the expected share price tends to diminish and disappear asymptotically. In the long run, the expected value of a stock is equal to the present value of the dividends.

And how is the evolution of the market price of the stock? With recourse to previous conditions, and some manipulation, we obtain the law of motion of the stock price.

$$\begin{cases} \dot{p}^e(t) = rp(t) - d \\ \dot{p}^e(t) = \alpha[p(t) - p^e(t)] \end{cases} \quad \Rightarrow \quad rp(t) - d = \alpha[p(t) - p^e(t)] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (r-\alpha)p(t) = d - \alpha p^e(t) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad p(t) = \frac{d}{r-\alpha} - \frac{\alpha}{r-\alpha}\frac{d}{r} - \frac{\alpha}{r-\alpha}\left[p^e(t_0) - \frac{d}{r} \right] e^{-\frac{r\alpha}{r-\alpha}(t-t_0)} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad p(t) = \frac{d}{r} - \frac{\alpha}{r-\alpha}\left[p^e(t_0) - \frac{d}{r} \right] e^{-\frac{r\alpha}{r-\alpha}(t-t_0)}.$$

The expression above shows that (with $\alpha < r$) the stock price also converges asymptotically to the fundamental value.

E. Rational Expectations

A disturbing feature of this model is that the forecast error is predictable by an agent that is able to make the calculations we made above. This presents an opportunity for quick profits. A way to avoid this criticism is to consider the more sophisticated formulation of rational expectations (Muth, 1961).

Since this model is purely deterministic, we use the much simpler notion of perfect-foresight.

$$p^e(t) = p(t). \qquad (8)$$

The dynamics of the system change drastically.

$$rp(t) = d + \dot{p}^e(t) = d - \dot{p}(t) \quad \Leftrightarrow \quad \dot{p}(t) = rp(t) - d. \qquad (9)$$

There is still a single steady-state solution.

$$0 = rp^* - d \qquad \Leftrightarrow \qquad p^* = \frac{d}{r}. \tag{10}$$

But now the system is unstable, because $r > 0$.

$$p(t) = p^* + [p(0) - p^*]e^{rt}. \tag{11}$$

The second term, that may be designated as the bubble, grows to plus or minus infinity unless the initial price equals the fundamental value of the stock. If we rule out these ever-growing bubbles, the only solution corresponds to share prices jumping instantaneously to the fundamental value of the stock.

Let's now introduce a tax rate, τ, on dividends and see how share prices reacts to changes in this tax rate. The equations must be modified to account for the fact that net dividends diminish to $(1-\tau)d$.

$$\dot{p}(t) = rp(t) - (1-\tau)d. \tag{12}$$

$$p(t) = p^* + [p(0) - p^*]e^{rt}. \tag{13}$$

$$p^* = \frac{(1-\tau)d}{r}. \tag{14}$$

An unexpected change in the tax rate, to a new value that is expected to remain forever, makes stock prices jump instantaneously to the new fundamental value.

A more interesting problem is to predict the movements of the stock prices in response to a credible announcement, at t_0 of a future change in the dividend tax rate, from τ_0 to τ_1, that will take place at t_1.

At t_1, the stock price will be at the new fundamental value, $p^*(\tau_1)$. For $t < t_1$, the law of motion derived for τ_0 applies. Now it is not reasonable to exclude the diverging solutions, because the divergence is only temporary.

$$p(t) = p^*(\tau_0) + [p(0) - p^*(\tau_0)]e^{rt}.$$

This expression must assume the value $p^*(\tau_1)$ at t_1. Otherwise, there would be a predictable jump in stock prices at t_1, allowing agents to make profits through arbitrage.

$$p^*(\tau_0) + [p(0) - p^*(\tau_0)]e^{rt_1} = p^*(\tau_1).$$

The stock price jump caused by the announcement is given below.

$$p(0) - p^*(\tau_0) = [p^*(\tau_1) - p^*(\tau_0)]e^{-rt_1}. \tag{15}$$

In sum, stock prices jump after the announcement, then diverge from the old fundamental value in a precise way in order to coincide with the new fundamental value when the tax change enters into effect. During the transition, stock prices are given by the following expression.

$$p(t) = p^*(\tau_0) + [p^*(\tau_0) - p^*(\tau_1)]e^{r(t-t_1)}.$$

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